Bifurications of gradient vector fields

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Chapter I. Introduction.

I.1 The Thom-Smale program.

Dynamical systems.

The subject of this thesis fits within the framework of differentiable dynamical systems, i.e. the theory of those mathematical objects which are used as models for the time-evolution of physical (chemical, biological) systems.
A dynamical system consists of a space M of possible states of the system and a vector field X ("differential equation") on M.
In case of a mechanical system, for instance, the points of M may represent position and velocity of a certain point mass. The vector field X represents the equations of motion, induced by an appropriate physical law (for instance Newton's second law).
Usually M may be considered as a subset of some Euclidian space, which has in a natural way the structure of a differentiable manifold. In the sequel we moreover assume that M is compact and of finite dimension, and that X is a smooth vector field on M. Integrating the vector field X, i.e. "solving the differential equation", we obtain a one-parameter group of smooth diffeomorphisms $\varphi_{X,t}: M \to M$, $t \in \mathbb{R}$, also called the flow of X, which describes the time evolution: a state of the system, which at time $t = 0$ corresponds to a point $x \in M$, will be represented by $\varphi_{X,t}(x) \in M$ after time t.

General vs. exceptional: Smale's program.

One of the main questions arising in mathematics is to describe the "essential" properties of the objects of study, and to find out how "general" objects with such a property are. A setting for this problem in the context of differentiable dynamical systems was set out by Smale in his survey paper [19]. In the context of the theory of dynamical systems one introduces a topology $\mathcal{T}$ on the space $\mathcal{X}(M)$ of all smooth systems on M, with the so called Baire-property: if $\{U_i\}_{i=1}^{\infty}$ is a collection of open and dense sets in $\mathcal{X}(M)$, then $\bigcap_{i=1}^{\infty} U_i$ is dense in $\mathcal{X}(M)$. A subset P of $\mathcal{X}(M)$ is then called residual if there is a countable collection of open and dense subsets $U_i \subset \mathcal{X}(M)$ with $P \supset \bigcap_{i=1}^{\infty} U_i$. A property is said to be generic if the set of systems, satisfying this property, is a redidual subset of $\mathcal{X}(M)$.
The concept of genericity is a widely accepted and criticized formalization of the notion "general" in everyday-language. It may be considered as the topological counterpart of the measure-theoretic concept of "full measure".
To describe "essential" properties a suitable equivalence relation usually is

introduced: it should be fine enough to distinguish obvious features (e.g. a system having two rest-points should not be identified with one having none at all), on the other hand it should be coarse enough to have rather large (e.g. open) equivalence classes. And, last but not least, one should be able to develop a satisfactory theory (i.e. to prove theorems) which really add something non-trivial to our knowledge.
In this respect especially the concept of stability is important; a system is called stable (with respect to an equivalence relation) if any system in a sufficiently small neighbourhood is equivalent to it. The most favoured would be an equivalence relation with respect to which the stable elements contain a dense open subset. If this proves impossible one might hope that stability occurs generically. Moreover, if a system is not stable, one should like to have a nice description of the equivalence classes in a neighbourhood of this system: preferably it should contain only a finite number of equivalence classes which are "neatly" arranged, or, if this is not possible, one should like them to be parameterized by a finite number of parameters (called "moduli").

Incorparating instabilities: the Thom-Smale program.

In stead of studying single dynamical systems, one may consider families of such objects. This is the situation one meets for instance when dealing with physical systems depending on external parameters.
Generically unstable systems may occur in such k-parameter families for certain parameter values, although the family itself may be stable (i.e. equivalent to any slightly perturbed family). By increasing the number of parameters we may expect to find instabilities with an "increasing degree of degeneracy", which will also be present in any sufficiently close family. This point of view was adopted by Thom in his celebrated book [12], when discussing models explaining discontinuous changes in both static and dynamic behaviour of physical and biological systems.

The program described above for single systems can be adapted slightly to incorporate the study of parametrized families. This extended program, sometimes called the Thom-Smale program, will be considered presently. First we touch it in the framework of the theory of real-valued functions on a manifold M, in which many beautiful results have been obtained, and in which moreover the situation is relatively well understood. Apart from the fact that these "systems" are easier to handle than vector fields, they are considered here because real-valued functions give rise to a special kind of dynamical systems which form our main interest: the so-called gradient systems.

Next we cast a glance at the state of affairs in dynamical systems concerning matters of stability: it has become apparent that there is no "all-purpose-equivalence-relation" in this context. Nevertheless the exploration of the world of dynamical systems has revealed many of its features. In search of stability one encountered much which now belongs to the common folklore, not in the last place owing to the picturesque name giving: whiskered tori, wild horse shoes, strange attractors, Ω-explosions, phantom kisses, blue-sky catastrophes,...